Int J Theor Phys (2008) 47: 2493–2499
DOI 10.1007/s10773-008-9682-5

Schwarzschild Solution on the Brane

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Received: 27 November 2007 / Accepted: 30 January 2008 / Published online: 6 February 2008

Abstract In this paper it is shown that Schwarzschild solution is possible in brane world for some specific choices of brane matter and the non-local effects from the bulk. A conformally flat bulk space time with fine-tuned vacuum energy (brane tension) shows that Schwarzschild solution may also be the vacuum solution for brane world scenario.

Keywords Matter field · Brane world · Schwarzschild solution

1 Introduction

In brane world paradigm, our universe is a slice of some higher dimensional space-time. In contrast to extra dimensional theory of Kaluza-Klein, it is possible to have large, even noncompact extra dimensions in brane world scenario. However, extra dimensions are unobservable for a brane observer at low energies while at higher energies, the extra dimensions may have experimental consequences [1–5]. Though the primary motivation of the phenomenology of brane world scenario is to provide a possible resolution of the hierarchy problem between weak and Planck scales, however, the cosmological and astrophysical implications of such scenario are also interesting from the observational point of view. In this context, Randall-Sundrum II brane world concept [4, 5], which describes our 4D world as a surface (a domain wall), supporting all standard matter fields and embedded in a 5D space-time (bulk), leads to a variety of models both in the cosmological context and in the description of local self gravitating objects. This idea is motivated by the Horava-Witten compactification of M-Theory [6, 7] and it consists of a domain wall universe (brane) embedded in 5D anti-de Sitter (Ads) space-time (bulk). Usually, specifying the dynamics and the geometry on the brane, one tries to extend the solution to the bulk. Though, it is not easy to find the bulk geometry with brane as its boundary and also in general the set of equations on the brane are not closed, this method is basically the only way to find non trivial brane world solutions.

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If the observed matter fields (uncharged and non-rotating) in brane world are trapped on the brane and undergo gravitational collapse, then the formed black hole ([8, 9] and for review see [10]) will be a higher dimensional object as it may have a horizon extending to the extra dimension. Phenomenological properties of such black holes have been studied by Argyres et al. [11] for RS II type brane scenario. The usual Schwarzschild-Ads solution represents a black hole localized in the fifth dimension but can not describe the final state of gravitational collapse on the brane. On the other hand, a black hole on the brane may be viewed as a black string in the higher dimensional Ads space-time. This is stable far from the Ads horizon, but becomes unstable near the Ads horizon. Chamblin et al. [12] have shown that it is a 'black cigar' solution that looks like black string far from the Ads horizon but has a horizon that closes off before reaching the Ads horizon.

Subsequently, Chamblin, Reall, Shinkai and Shiromizu [13] have studied charged brane black hole. They have shown that such black holes may have two types of 'charge'—one from a Maxwell field trapped on the wall and other arising from the bulk Weyl tensor. They have found a Reissner-Nordström solution (as obtained in [14]) as induced metric on the brane in the absence of Maxwell charge while by inducing the Maxwell charge the geometry is that of Reissner-Nordström with small corrections. They have used a combination of analytical and numerical techniques to study how these black holes behave in the bulk.

In the present paper, we have studied brane black hole in a different way. We have examined the possible matter source in the brane if the induced brane black hole is assumed to be Schwarzschild. Using the idea of Birkhoff's theorem, the matter field in the domain wall (brane) is such that the effective energy-momentum tensor is zero on the brane. We have shown that a class of solutions is possible for matter distribution on the brane which results Schwarzschild black hole on the brane. Finally, we have presented some possible choices for the matter distribution and their properties are discussed.

2 Brane-Bulk Geometry: The Basic Equations

Let us consider a four dimensional hypersurface Σ—the three brane (thin domain wall) embedded in a 5-dimensional space-time $\mathcal{M}$ (bulk). In general, we choose a coordinate y such that the hypersurface $y=0$ coincides with the brane. The total action for the system is taken as

$$\begin{aligned} S = &\frac{1}{2\kappa_5^2}\int_{\mathcal{M}} \sqrt{-{}^{(5)}g}\left({}^{(5)}R - 2\Lambda_5\right) d^5x + \frac{1}{2\kappa_4^2}\int_{\Sigma} \sqrt{-{}^{(4)}g}\left({}^{(4)}R - 2\Lambda_4\right) d^4x \\ &+ \int_{\mathcal{M}} \sqrt{-{}^{(5)}g}\, L_5^{matter} d^5x + \int_{\Sigma} \sqrt{-{}^{(4)}g}\, L_4^{matter} d^4x \end{aligned} \tag{1}$$

Here the dimensionful constants κ_5^2 and κ_4^2 are related to the Planck masses M_5 and M_4 by the relations

$$\kappa_5^2 = 8\pi G_{(5)} = M_5^{-3}, \quad \kappa_4^2 = 8\pi G_{(4)} = M_4^{-2} \tag{2}$$

and Λ_4/κ_4^2 can be interpreted as the brane tension λ of the standard Dirac-Nambu-Goto action. From string theory point of view, we concern with vacuum bulk (i.e., ${}^{(5)}T_{AB}=0$). Now varying equation (1) with respect to the bulk metric, we obtain the five dimensional Einstein equation

$${}^{(5)}G_{AB} = \kappa_5^2\left[-\Lambda_5^{(5)} g_{AB} + \delta(y)\left(-\lambda^{(4)} g_{AB} + {}^{(4)}T_{AB}\right)\right] \tag{3}$$

where ${}^{(4)}g_{AB}$ is the induced metric on the wall, given by

$$ {}^{(4)}g_{AB} = {}^{(5)}g_{AB} - n_A n_B \tag{4} $$

with n_A, the unit normal to the wall.

If we assume the bulk space-time to be symmetric under reflections in the wall, then $y = 0$ is the fixed point of the Z_2-reflection symmetry. Due to the singular source at $y = 0$, the modified Israel-Darmois-Lanczos-Sen condition [15–19] gives

$$ K_{\mu\nu}|_{y=0} = -\frac{1}{6}\kappa_5^2 \lambda^{(4)} g_{\mu\nu} - \frac{1}{2}\kappa_5^2 \left({}^{(4)}T_{\mu\nu} - \frac{1}{3}{}^{(4)}g_{\mu\nu}^{(4)} T \right) \tag{5} $$

where $K_{\mu\nu} = \frac{1}{2}\partial y^{(4)} g_{\mu\nu}$ is the extrinsic curvature of the domain wall. Here the extrinsic curvature is calculated on the side of the domain wall that the normal points into. The reason is that, we want to evolve the initial data prescribed on the wall in the direction of this normal. Using the Gauss and Codazzi equations [20] and the above junction condition, the effective Einstein equation on the brane [21–23] is

$$ {}^{(4)}G_{\mu\nu} = -\Lambda_4^{(4)} g_{\mu\nu} + \kappa_4^{2(4)} T_{\mu\nu} + \frac{6\kappa_4^2}{\lambda}\Pi_{\mu\nu} - E_{\mu\nu} \tag{6} $$

where $\Lambda_4 = \frac{1}{2}\kappa_5^2(\Lambda_5 + \frac{1}{6}\kappa_5^2\lambda^2)$.

Compared to Einstein's general relativity, there are two additional terms in the energy-momentum tensor due to embedding of the brane into the bulk. The term $\Pi_{\mu\nu}$ is quadratic in ${}^{(4)}T_{\mu\nu}$ arising from the extrinsic curvature terms in the projected Einstein tensor and is given by

$$ 4\Pi_{\mu\nu} = \frac{1}{3}{}^{(4)}T^{(4)}T_{\mu\nu} - {}^{(4)}T_{\mu\alpha}^{(4)}T_\nu^\alpha + \frac{1}{2}{}^{(4)}g_{\mu\nu}\left({}^{(4)}T_{\alpha\beta}^{(4)}T^{\alpha\beta} - \frac{1}{3}{}^{(4)}T^2 \right) \tag{7} $$

This is known as local bulk effect. The second correction term $E_{\mu\nu}$ stands for non-local bulk effect and is the projection of the five dimensional Weyl tensor ${}^{(5)}C_{ABCD}$ onto the brane *viz.* [21]

$$ E_{\mu\nu} = {}^{(5)}C_{ABCD}\delta_\mu^A \delta_\nu^C n^B n^D \tag{8} $$

Now the induced metric on the brane is assumed to be in the form

$$ ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega_2^2 \tag{9} $$

with $A(r) = 1 - \frac{2M}{r}$.

Then the bulk metric can be chosen in the form [13]

$$ ds^2 = N^2(y,r)dy^2 - e^{2\alpha(y,r)}A(r)dt^2 + \frac{e^{2\beta(y,r)}}{A(r)}dr^2 + e^{2\delta(y,r)}r^2 d\Omega_2^2 \tag{10} $$

The lapse function N describes the embedding geometry of the hypersurface $y = 0$ during the evolution in the bulk space-time.

The components of the extrinsic curvature of the hypersurface $y = 0$ having unit normal $n = N dy$, are given by [13]

$$ K_t^t = \frac{\dot{\alpha}}{N}, \quad K_r^r = \frac{\dot{\beta}}{N}, \quad \text{and} \quad K_\theta^\theta = K_\phi^\phi = \frac{\dot{\delta}}{N} \tag{11} $$

where an overdot stands for differentiation with respect to y. Then the space-time is described by the evolution equation

$$\dot{K}^{\mu}_{\nu} = N\left({}^{(4)}R^{\mu}_{\nu} - KK^{\mu}_{\nu} + \frac{4}{l^2}\delta^{\mu}_{\nu}\right) - D^{\mu}D_{\nu}N \tag{12}$$

together with the constraint equations

$${}^{(4)}R - K^2 + K_{\mu\nu}K^{\mu\nu} = -\frac{12}{l^2} \quad \text{(scalar constraint)} \tag{13}$$

and

$$D_{\mu}K^{\mu}_{\nu} - D_{\nu}K = 0 \quad \text{(vector constraint)} \tag{14}$$

Here ${}^{(4)}R^{\mu}_{\nu}$ and ${}^{(4)}R$ are the usual Ricci tensor and Ricci scalar on the hypersurface $y = 0$ and D_{μ} is the covariant derivative operator on the hypersurface.

3 Schwarzschild Brane and Possible Matter Field

When matter on the brane is absent (i.e, ${}^{(4)}T_{\mu\nu} \equiv 0$) and the brane tension is fine tuned so that, $\Lambda_4 = 0$, then the above modified Einstein equation (1) simplifies to

$${}^{(4)}G_{\mu\nu} = -E_{\mu\nu} \tag{15}$$

This is termed as vacuum Einstein equation on the brane. Subsequently, Dadhich et al. [14] has obtained Reissner-Nordström black hole solution but without electric charge being present. It is interpreted as tidal charge, arising from the projection onto the brane of free gravitational field effects in the bulk. Vollick [24] has obtained some solutions solving ${}^{(4)}R = 0$. He speculated that the Weyl term could be responsible for the observed dark matter in the universe. Then Bronnikov and Kim [25] have presented static, spherically symmetric Lorentzian wormhole solutions by solving ${}^{(4)}R = 0$.

In the present work, we consider a static spherically symmetric brane model with isotropic fluid confined in the brane. So, in an orthonormal reference frame, the form of the energy momentum tensor is

$${}^{(4)}T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p) \tag{16}$$

Further, due to static spherically symmetric nature of the problem, the projected Weyl tensor has the form [26]

$$E_{\mu\nu} = \mathrm{diag}[\epsilon(r), \sigma_r(r), \sigma_t(r), \sigma_t(r)] \tag{17}$$

Then the modified Einstein equations on the brane are

$$\left.\begin{aligned} {}^{(4)}G_{tt} &= 8\pi\ G\rho^{eff} \\ {}^{(4)}G_{rr} &= 8\pi\ Gp_r^{eff} \\ {}^{(4)}G_{\theta\theta} &= 8\pi\, Gp_t^{eff} \end{aligned}\right\} \tag{18}$$

where

$$\left.\begin{aligned} \rho^{eff} &= \rho\left(1+\tfrac{\rho}{2\lambda}\right)-\tfrac{\epsilon}{8\pi} \\ p_r^{eff} &= p\left(1+\tfrac{\rho}{\lambda}\right)+\tfrac{\rho^2}{2\lambda}-\tfrac{\sigma_r}{8\pi} \\ p_t^{eff} &= p\left(1+\tfrac{\rho}{\lambda}\right)+\tfrac{\rho^2}{2\lambda}-\tfrac{\sigma_t}{8\pi} \end{aligned}\right\} \tag{19}$$

Now if the static spherically symmetric space-time in the brane is chosen to be Schwarzschild, then $G_{tt}=0=G_{rr}=G_{\theta\theta}$ and consequently we have

$$\left.\begin{aligned} &\rho\left(1+\tfrac{\rho}{2\lambda}\right)-\tfrac{\epsilon}{8\pi}=0 \\ &p\left(1+\tfrac{\rho}{\lambda}\right)+\tfrac{\rho^2}{2\lambda}-\tfrac{\sigma_r}{8\pi}=0 \\ &p\left(1+\tfrac{\rho}{\lambda}\right)+\tfrac{\rho^2}{2\lambda}-\tfrac{\sigma_t}{8\pi}=0 \end{aligned}\right\} \tag{20}$$

with

$$-\epsilon+\sigma_r+2\sigma_t=0 \tag{21}$$

However, from the second and third equations of (20), one gets $\sigma_r=\sigma_t(=\sigma)$ and hence essentially we have the following two equations containing three unknowns

$$\left.\begin{aligned} &\rho\left(1+\tfrac{\rho}{2\lambda}\right)-\tfrac{3\sigma}{8\pi}=0 \\ \text{and}\quad &p\left(1+\tfrac{\rho}{\lambda}\right)+\tfrac{\rho^2}{2\lambda}-\tfrac{\sigma}{8\pi}=0 \end{aligned}\right\} \tag{22}$$

Thus we have one parameter family of solutions which describes matter field on the brane to have induced Schwarzschild black hole.

We shall now present solutions for some typical choices of the matter field:

Case-I: $\sigma=0$
Then we have two possibilities

$$\rho=0,\ p=0 \quad \text{or} \quad \rho=-2\lambda,\ p=2\lambda$$

Thus Schwarzschild solution is possible for either vacuum brane or brane with strange matter distribution ($\rho<0$ and $p>0$) embedded in a conformally flat bulk. The Schwarzschild black hole solution in the brane with negative matter density is supported by Vollick [27]. This 4D black hole from the point of view of an observer in the brane world is essentially a black string (black cigar) in five dimensional Schwarzschild Ads bulk [12]. The other choice (vacuum solution) is not of much interest. Chamblin et al. [12] have shown that 5D Schwarzschild-Ads bulk can not intersect vacuum domain wall and hence it is not a possible final state of gravitational collapse on the brane.

Case-II: $p=0$ (*Dust*)
Then we have $\rho=\lambda$ and $\sigma=4\pi\lambda$

So it is possible to have Schwarzschild solution for dust brane with non-local correction term behaving as perfect fluid with radiation equation of state.

Case-III: $p=\omega\rho$ (*Perfect fluid*)
The expressions for ρ and σ are given by

$$\rho=\lambda\left(\frac{1-3\omega}{1+3\omega}\right)$$

and

$$\sigma = 4\pi\lambda \frac{(1-3\omega)(1+\omega)}{(1+3\omega)^2}$$

with $-1/3 < \omega < 1/3$.

So, a realistic perfect fluid with linear equation of state has Schwarzschild solution provided that the non-local bulk effect also behaves as a perfect fluid. In both case II and case III, we have non-zero bulk Weyl tensor which behaves as perfect fluid with radiation equation of state. Note that in vacuum brane, this non-local bulk contribution gives rise to a tidal charge and there is Reissner-Nordström black hole on the brane [14]. But the matter in the brane balances this non-local bulk effect and we have Schwarzschild black hole in the brane. Also in these cases, the induced black hole in the brane appears from the bulk as black string, which may have some properties different from the previous case.

Further, we note that the non-local bulk effect as expressed by the projection of the Weyl tensor on the brane, does not make the brane dynamics closed. Usually, the geometry of the bulk is required to deal with the dynamics on the brane. In the present paper, we have imposed assumption either on the Weyl tensor or on the brane matter to make the system closed.

Therefore by fixing the geometry of the brane world, it is possible to determine various possibilities for the matter to have prescribed geometric structure. Usually in the literature, matter is prescribed and we have to determine the geometry by solving second order non linear differential equations. On the other hand, in the present work, we have fixed the geometry and matter is determined by solving a set of algebraic equations. So we may conclude that, our method is more suitable than the existing standard approach. Lastly, we conclude that Birkhoff's theorem for unique spherically symmetric solution is not possible in brane scenario.

Acknowledgements The work has been done during a visit to IUCAA under the associateship programme. The authors gratefully acknowledge the warm hospitality and facilities of work at IUCAA. Also T.B is thankful to CSIR, Govt. of India, for awarding Junior Research Fellowship.

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